Slowly varying nonlinear waves in a warm plasma stream

1981 J. Phys. A: Math. Gen. 14 2113

(http://iopscience.iop.org/0305-4470/14/8/032)

J. Phys. A: Math. Gen. **14** (1981) 2113–2119. Printed in Great Britain

Slowly varying nonlinear waves in a warm plasma stream

R J Gribben and E J Parkes

Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, UK

Received 8 September 1980, in final form 5 February 1981

Abstract. This paper extends previous work on the propagation of one-dimensional nonlinear waves in a cold, collisionless, field-free, slightly non-uniform plasma to the case where the plasma is warm. The formulation is as before and the wave modulation properties are discussed; here, however, it is necessary to solve an initial value problem for the modulation equations by an approximate technique.

1. Introduction

In this paper we consider a similar physical situation to that treated previously (Gribben and Parkes 1977, hereafter referred to as I), but we now include a pressure proportional to n^3, where n is the electron number density. This mild change in the basic equations increases enormously the difficulties in their solution.

The basic uniform solution (essentially that given by Coffey (1971)) presented in § 2 takes the form of an expansion in amplitude and assumes that no particles are trapped. This condition imposes a restriction on amplitude that grows more severe as the temperature parameter $\beta (0 \leqslant \beta < 1)$ increases from zero.

For the non-uniform plasma considered in § 3, the modulation equations are derived as before, but the method now requires that departures from uniformity (measured by ε) are of smaller order than that of the amplitude. There are four amplitude-dependent families of characteristics of the equations, compared with a single family independent of the amplitude in the cold plasma. Two of the characteristic speeds coalesce in the linear limit to the linear group velocity.

In § 4 the strained coordinate method (Parkes and Gribben 1978) is used to solve an initial value problem for the modulation equations. Of the choice of four characteristic coordinates to strain, the most suitable seems to be the pair which remain distinct at small amplitudes. The solution is carried out to include the leading nonlinear effects.

The principal features of this solution are described in § 5 for a finite, initially symmetric amplitude distribution. Whereas, in the cold plasma, the amplitude profile propagates with distortion in a symmetric manner relative to the peak amplitude, here there is an initial period of asymmetric distortion, although eventually the distortion is symmetric.

2. Uniform waves

With the notation used in I, the basic equations for the problem can be written as

$$\partial n/\partial t + \partial (nv)/\partial x = 0, \tag{1}$$

0305-4470/81/082113+07\$01.50

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}\left(\frac{v^2}{2}+\frac{3cn^2}{2m}-\frac{e\phi}{m}\right)=0, \tag{2}$$

$$\frac{\partial}{\partial t}(nv)+\frac{\partial}{\partial x}\left[nv^2+\frac{cn^3}{m}-\frac{\varepsilon_0}{2m}\left(\frac{\partial\phi}{\partial x}\right)^2\right]-\frac{Ne}{m}\frac{\partial\phi}{\partial x}=0. \tag{3}$$

The constant c is zero for the cold plasma considered in I, and here, for a finite electron temperature, $c=pn^{-3}$ is the adiabatic pressure law $p\propto n^{\gamma}$ with $\gamma=3$, where p is the pressure.

Equations (1)–(3) are valid if the wavelength of a wave motion is large compared with the Debye length (Bernstein and Trehan 1960), and they also describe the 'single water bag' model of a plasma (Davidson 1972, § 3.5).

Uniform nonlinear 2π-periodic solutions of (1)–(3) are obtained by transforming to the wave frame as in I, but here it is necessary to resort to expansions in amplitude. With boundary conditions $u=V_0$, $n=N$ and $\phi=0$ at $\chi=0$ we find (cf Coffey 1971) that

$$\begin{aligned} e\phi/mV_0^2&=-a_*\beta_1\sin\chi+a_*^2(3-4\cos\chi+\cos 2\chi)\beta_2/12+\mathrm{O}(a_*^3),\\ n/N&=1+a_*\sin\chi+a_*^2(\cos\chi-\cos 2\chi)\beta_2/3\beta_1+\mathrm{O}(a_*^3),\\ u/V_0&=1-a_*\sin\chi+a_*^2[3\beta_1-2\beta_2\cos\chi+(2\beta_2-3\beta_1)\cos 2\chi]/6\beta_1+\mathrm{O}(a_*^3),\end{aligned} \tag{4}$$

where u and χ are as in I, $a_*=a/(1-\beta)^{1/2}$, $\beta_1=1-\beta$, $\beta_2=3+\beta$, $\beta=3V_{\mathrm{th}}^2/V_0^2=3cN^2/mV_0^2$, where V_{th} is the undisturbed electron thermal velocity and $\beta<1$ for periodic waves. Coffey (1971) showed that for trapped particles to be excluded

$$a^2<1-\beta/3-8\beta^{1/4}/3+2\beta^{1/2}, \tag{5}$$

where

$$a=-\frac{e}{mV_0\omega_{\mathrm{p}}}\left[\frac{\partial\phi}{\partial x}\right]_{\chi=0},$$

$\omega_{\mathrm{p}}^2=Ne^2/m\varepsilon_0$ and $|a|$ is a dimensionless measure of the maximum amplitude of $-\partial\phi/\partial x$, the electric field.

The condition (5) implies that the assumed model is only valid for small amplitudes. For example, with $\beta=0.5$, (5) gives $|a|<0.0718$. Hence the small-amplitude solution (4) is appropriate.

For there to be no secular terms in the solution, the wavenumber k and the frequency ω satisfy

$$(\omega-kU)^2=\omega_{\mathrm{p}}^2+3k^2V_{\mathrm{th}}^2+a_*^2\omega_{\mathrm{p}}^2\beta(\beta+15)/6\beta_1^2+\mathrm{O}(a_*^4), \tag{6}$$

where $U=V+V_0$ and $V=\omega/k$. To leading order (6) is the Doppler shifted form of the familiar linear dispersion relation for long-wavelength longitudinal oscillations in a warm plasma. Unlike the cold plasma stream (I), the nonlinear dispersion relation is amplitude dependent and the waves genuinely propagate through the plasma since the linear group velocity is $\partial\omega/\partial k=U-V_0\beta$.

We now go on to consider the equations satisfied by a, N, V_0, U, ω and k when we allow these quantities to vary slowly in the sense that significant changes can occur over a large number of wavelengths.

3. The modulation equations and characteristic velocities

The formalism for dealing with the propagation of waves through a slightly non-uniform plasma using the uniform solution has been described before (see I). In the present case the averaged equations, to first order in A, corresponding to (1)–(3) are found to be

$$\partial N/\partial T + \partial(NU)/\partial X = 0, \tag{7}$$

$$\frac{\partial}{\partial T}(U + \tfrac{1}{2}A^* V_0) + \frac{\partial}{\partial X}[\tfrac{1}{2}U^2 + \tfrac{1}{2}\beta V_0^2 + \tfrac{1}{2}A^* V_0(U - V_0)] = 0, \tag{8}$$

$$\frac{\partial}{\partial T}(NU) + \frac{\partial}{\partial X}[NU^2 + \tfrac{1}{4}(1+3\beta)A^* N V_0^2 + \tfrac{1}{3}\beta N V_0^2] - N\frac{\partial}{\partial X}(\tfrac{1}{4}\beta_2 A^* V_0^2) = 0, \tag{9}$$

where $A^* = a_*^2$, and for consistency $A = a^2$ must be of larger order of magnitude than the parameter ε which measures the ratio of a typical period or wavelength to a typical time scale or length scale of the modulation. The compatibility condition is

$$\partial k/\partial T + \partial \omega/\partial X = 0 \tag{10}$$

and ω and k are related by $\omega = k(U - V_0)$ and (6). Equation (9), which is not in conservation form, may be replaced by

$$\frac{\partial}{\partial T}(A^* N^{1/2} V_0^2 \beta_1^{1/2}) + \frac{\partial}{\partial X}[(U - V_0\beta)A^* N^{1/2} V_0^2 \beta_1^{1/2}] = 0. \tag{11}$$

To obtain (11), some algebraic manipulation is required, but it can also be derived directly using Whitham's averaged Lagrangian method (see Parkes 1980), and can be identified as the 'conservation of wave action' equation.

When $\beta = 0$, (6)–(9) and (11) reduce to the cold plasma evolutionary equations, (3.6)–(3.9) and (3.11), valid for $0 < A < 1$, of I; here they are subject to the more restrictive condition $0 < A \ll 1$.

The characteristic velocities of the hyperbolic system of equations are obtained by standard methods, although the procedure is considerably more tedious than in the corresponding cold plasma case. Writing $\mathrm{d}X/\mathrm{d}T = C = U + V_0 Y$, we obtain

$$(Y+\beta)^2(Y^2-\beta) + A\beta(Y+1)\{p + q(Y^2-\beta) + (Y+\beta)[r + s(Y^2-\beta)]\} = \mathrm{O}(A^2), \tag{12}$$

where $p = (1+3\beta)/4$, $q = -\beta(\beta+15)/12\beta_1^2$, $r = (3\beta-7)/4\beta_1$ and $s = (-\beta^2 + 20\beta + 45)/12\beta_1^3$. When $\beta = 0$, (12) collapses to $Y^4 = 0$, corresponding to the quadruple characteristic velocity $C = U$ valid for all A in $0 < A < 1$ (see I). To leading order in A for $\beta \neq 0$, $(Y+\beta)^2(Y^2-\beta) = 0$ and the linear characteristic velocities are $U - V_0\beta$ (twice) and $U \pm V_0\beta^{1/2}$, the coincident pair being the linear group velocity. If we fix β and take A sufficiently small these velocities, to next order, are

$$\begin{aligned} C_\pm &= U - V_0\beta \pm a V_0 Y_1 + \mathrm{O}(a^2), \\ \mathscr{C}_\pm &= U \pm V_0\beta^{1/2} + A V_0 Y_\pm + \ldots, \end{aligned} \tag{13}$$

where $Y_1^2 = (3 + 6\beta + 6\beta^2 + \beta^3)/12\beta_1$ and $Y_\pm = [8\beta^{1/2} \mp (1+3\beta)]/8\beta_1\beta^{1/2}$. The double characteristic velocity of linear theory splits into two distinct real velocities C_+ and C_-. Whitham (1974, § 15.4) has suggested that such velocities be taken as the nonlinear group velocities. By contrast, there is no splitting in the cold plasma case because the nonlinear dispersion relation is amplitude independent. However, it remains true here that there is no instability of the type associated with equations having an elliptic part. Hence slowly varying small perturbations on the uniform periodic solution do not grow exponentially with time (cf Infeld and Rowlands 1979).

The characteristic velocities (13) were obtained on the assumption of fixed β and sufficiently small A. Henceforth, therefore, it will not be possible to deduce corresponding cold plasma results as the limiting case $\beta \to 0$.

4. Solution of the modulation equations

We consider an initial problem for (7)–(10) satisfying

$$N = \hat{N}, \qquad V_0 = \hat{V}_0(>0), \qquad U = \hat{U}, \qquad A = \mu \hat{A}(X) \text{ at } T = 0,$$

where $\hat{N}$, $\hat{V}_0$ and $\hat{U}$ are constant and μ is the initial maximum value of A. From (5) μ satisfies

$$\mu < 1 - \hat{\beta}/3 - 8\hat{\beta}^{1/4}/3 + 2\hat{\beta}^{1/2}, \tag{14}$$

where $\hat{\beta}$ is the initial value of β.

Unlike the cold plasma problem, where an exact solution of the governing equations was found, we are here compelled to resort to an approximate solution, and since (14) implies that μ is small (e.g. if $\hat{\beta} = 0.5$, $\mu < 0.005\,16$) the strained coordinate technique based on small μ is appropriate (see Nayfeh 1973, § 3.2). For consistency $\mu \gg \varepsilon$ (see § 3).

We note first from (13) that the families of linear characteristics of the equations are

$$\xi_0 = X - (\hat{U} - \hat{V}_0\hat{\beta})T = \text{constant}, \tag{15}$$

$$\phi_{0\pm} = X - (\hat{U} \pm \hat{V}_0\hat{\beta}^{1/2})T = \text{constant}. \tag{16}$$

Though it is not obvious, it turns out to be most convenient to use characteristic variables, ϕ_+, ϕ_-, given by

$$\frac{\partial X}{\partial \phi_-} = \mathscr{C}_+ \frac{\partial T}{\partial \phi_-}, \qquad \frac{\partial X}{\partial \phi_+} = \mathscr{C}_- \frac{\partial T}{\partial \phi_+}, \tag{17}$$

in place of the independent variables X, T. The coordinate straining is determined by new dependent variables X, T, which, together with all other dependent variables, are expanded in powers of μ in the form

$$A = \mu A^{(0)}(\phi_+, \phi_-) + \ldots,$$
$$X = X^{(0)}(\phi_+, \phi_-) + \mu X^{(1)}(\phi_+, \phi_-) + \ldots.$$

Substitution of these series into the equations (7)–(10), (17) and the initial conditions applied at $\phi_+ = \phi_- = X$ yields a hierarchy of equations for the coefficients.

Leading-order solution

We find that $N^{(0)} = \hat{N}$, $U^{(0)} = \hat{U}$, $V_0^{(0)} = \hat{V}_0$ and $A^{(0)} = \hat{A}(\xi)$, where

$$\xi(\phi_+, \phi_-) = X^{(0)} - (\hat{U} - \hat{V}_0\hat{\beta})T^{(0)}. \tag{18}$$

Thus, to leading order, $\xi = \text{constant}$ is (15), the double characteristic corresponding to the linear group velocity $\hat{U}-\hat{V}_0\hat{\beta}$. From (17) it follows that

$$X^{(0)}=[(\hat{U}+\hat{V}_0\hat{\beta}^{1/2})\phi_- -(\hat{U}-\hat{V}_0\hat{\beta}^{1/2})\phi_+]/2\hat{V}_0\hat{\beta}^{1/2},$$
$$T^{(0)}=(\phi_- -\phi_+)/2\hat{V}_0\hat{\beta}^{1/2}.$$

First-order solution

The corresponding results for the first-order functions are found to be

$$\begin{aligned}
N^{(1)}&=\hat{N}[2\hat{A}(\xi)-\nu_-\hat{A}(\phi_+)-\nu_+\hat{A}(\phi_-)]/4\hat{\beta}_1\hat{\beta},\\
U^{(1)}&=\hat{V}_0[\nu_+\hat{A}(\phi_-)-\nu_-\hat{A}(\phi_+)-2\hat{\beta}^{1/2}\hat{A}(\xi)]/4\hat{\beta}_1\hat{\beta}^{1/2},\\
V_0^{(1)}&=(1+3\hat{\beta})U^{(1)}/2\hat{\beta}-\hat{V}_0N^{(1)}/\hat{N}-(\phi_- -\phi_+)\hat{V}_0\hat{Y}_1^2\hat{A}'(\xi)/2\hat{\beta}^{3/2},\\
X^{(1)}&=\hat{U}(R_+ + R_-)/\hat{V}_0+(R_+ - R_-)\hat{\beta}^{1/2}, \qquad T^{(1)}=(R_+ + R_-)/\hat{V}_0,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
R_\pm&=\nu_\pm(\phi_- -\phi_+)\hat{A}(\phi_\mp)/8\hat{\beta}_1\hat{\beta}^{3/2}\pm P_\pm[F(\xi)-F(\phi_\mp)],\\
P_\pm&=(3\mp4\hat{\beta}^{1/2}-3\hat{\beta})/16\nu_\mp\hat{\beta}_1\hat{\beta}^{3/2}, \qquad \nu_\pm=1\pm\hat{\beta}^{1/2},\\
F(z)&=\int_0^z \hat{A}(s)\,\mathrm{d}s.
\end{aligned}$$

$A^{(1)}$ is not calculated since it involves terms of $\mathrm{O}(a_*^4)$ in the uniform solution.

5. Discussion

First we note that the solution for a cold plasma for A, and for example N, is (Parkes and Gribben 1978)

$$A=\mu\hat{A}(\xi), \qquad N=\hat{N}+\tfrac{1}{4}\mu\hat{N}\hat{V}_0^2T^2\hat{A}''(\xi),$$

where

$$\xi=\xi_0-\tfrac{1}{4}\mu\hat{V}_0^2T^2\hat{A}'(\xi), \qquad \xi_0=X-\hat{U}T.$$

For a symmetric initial profile the mid-point travels at the uniform velocity $\hat{U}$ in the laboratory frame, and the distortion relative to the mid-point is symmetric for all $T>0$.

For a warm plasma we also consider trajectories of disturbances in the ξ_0-T plane (but with $\xi_0=X-(\hat{U}-\hat{V}_0\hat{\beta})T$), i.e. we work in a reference frame moving with the constant linear group velocity relative to the laboratory frame. In this plane the straight characteristics (16) have slopes $(\pm\nu_\pm\hat{V}_0\hat{\beta}^{1/2})^{-1}$ respectively. Linear theory predicts that A propagates without distortion along the straight lines $\xi_0=$ constant, whereas our corrected approximate solution $A=\mu\hat{A}(\xi)$, where from (18) and (19)

$$\xi=\xi_0-\mu[(R_+ - R_-)\hat{\beta}^{1/2}+(R_+ + R_-)\hat{\beta}], \tag{20}$$

says that A propagates along the curved lines $\xi=$ constant so that the A profile becomes distorted.

If we localise the initial profile, so that $\hat{A}(X)\neq0$ only in $-L/2<X<L/2$, we can say more about this distortion. Since $\xi=\xi_0=X$ at $T=0$, the profile propagates so that $A\neq0$ only in the region $D=\{(\xi_0,T): -L/2<\xi<L/2,\ T>0\}$ in the ξ_0–T plane. Now of all the straight lines $\phi_{0\pm} = \text{constant}$ within $D_0 = \{(\xi_0, T): -L/2 < \xi_0 < L/2, T > 0\}$, those entering at $T \geq T_-$, where $\hat{V}_0 T_-/L = (\nu_- \hat{\beta}^{1/2})^{-1}$, originated outside D_0. Consequently, allowing for the slight curvature of the $\phi_\pm$ characteristics and of the boundaries $\xi = \pm L/2$ of the A profile, we expect that for $T \geq T_-$ all $\phi_\pm$ characteristics within D originated outside D. Hence, within D, $\hat{A}(\phi_\pm) = 0$ and $F(\phi_\pm)$ are constant, so R_+ and R_- depend eventually on ξ only and the curves $\xi = \text{constant}$ become straight lines parallel to, and displaced from, the linear characteristics $\xi_0 = \text{constant}$. Thereafter, to first order in μ, the distorted A profile propagates without further change. A measure of the distortion from the initial profile is $\xi_0 - \xi$, tabulated in table 1 for $\hat{\beta} = 0.5$ for the symmetric profile

$$\hat{A}(X) = \begin{cases} [4(0.5+X)(0.5-X)/L^2]^4, & -L/2 < X < L/2, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Table 1. The values of $(\xi_0 - \xi) \times 10^6$ are given, calculated from (20), for the initial profile given by (21), with $\mu = 0.005$, $\hat{\beta} = 0.5$ and $\hat{U}/\hat{V}_0 = 1.25$. $d(\xi)$ is calculated from (22).

$A(\xi) \times 10^6$	7	839	3 429	5 000	3 429	839	7
$\hat{V}_0 T/L$ \\ ξ	−0.45	−0.30	−0.15	0.00	0.15	0.30	0.45
1	3 690	9 857	11 601	6 662	1 562	447	377
2	21 506	19 610	8 802	2 354	988	447	377
3	22 453	7 560	3 327	2 155	988	447	377
4	5 331	3 863	3 322	2 155	988	447	377
5	3 933	3 863	3 322	2 155	988	447	377
$d(\xi) \times 10^6$	1 778	1 708	1 167	0	−1 167	−1 708	−1 778

Here $\hat{V}_0 T_-/L = 4.83$ and so the table entries for $\hat{V}_0 T_-/L = 5$ give the final distortion of the A profile to this order. Thus in the early stages the distortion is asymmetric and greatest at the trailing edge. Relative to the peak of the profile, the final distortion is measured by

$$d(\xi) \equiv \xi_0 - \xi - \xi_{0\mathrm{p}} = \mu[(P_+ + P_-)\hat{\beta}^{1/2} + (P_+ - P_-)\hat{\beta}]F(\xi), \tag{22}$$

where $\xi_0 = \xi_{0\mathrm{p}}$ is the final trajectory of the peak and $\xi_{0\mathrm{p}} \equiv \xi_0(\xi = 0)$. Since (21) is symmetric, $d(-\xi) = -d(\xi)$, the final distortion is symmetric and A propagates without further distortion at velocity $\hat{U} - \hat{V}_0\hat{\beta}$ in the laboratory frame. Note that for $\xi > 0$, $d(\xi) \lesseqgtr 0$ for $\hat{\beta} \gtreqless \frac{1}{3}$ respectively.

The contributions to $N^{(1)}$ and $U^{(1)}$ proportional to $\hat{A}(\xi)$ behave as above, but those proportional to $\hat{A}(\phi_+)$ and $\hat{A}(\phi_-)$ propagate to right and left respectively along characteristics $\phi_+ = \text{constant}$, $\phi_- = \text{constant}$, which are curved, and so distortion occurs. For an initial profile like (21), distortion persists even for times beyond that for which $\hat{A}(\phi_+)$ and $\hat{A}(\phi_-)$ have any effect in D. In the case of $V_0^{(1)}$, propagation without distortion in D is never attained because of the term $\hat{A}'(\xi)$.

Finally, it is evident that the $\phi_\pm$ characteristics provide the nonlinear coupling between the wavetrain and changes in the background quantities N, V_0 and U (cf Whitham 1974, § 15.2).

References

Bernstein I B and Trehan S K 1960 *Nucl. Fusion* **1** 3
Coffey T P 1971 *Phys. Fluids* **14** 1402
Davidson R C 1972 *Methods in Nonlinear Plasma Theory* (New York: Academic)
Gribben R J and Parkes E J 1977 *J. Plasma Phys.* **18** 495
Infeld E and Rowlands G 1979 *J. Phys. A: Math. Gen.* **12** 2255
Nayfeh A H 1973 *Perturbation Methods* (New York: Wiley–Interscience)
Parkes E J 1980 *Mathematical Methods of Plasmaphysics* ed. R Kress and J Wick (Frankfurt: Peter D Lang) pp 37–49
Parkes E J and Gribben R J 1978 *J. Phys. A: Math. Gen.* **11** 2341
Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley–Interscience)